Journal of Applied Mechanics and Technical Physics, Vol. 42, No. 5, pp. 898–901, 2001

EQUATIONS OF FINITE BENDING OF THIN-WALLED CURVILINEAR TUBES

S. V. Levyakov

UDC 539.3

A system of nonlinear equations proposed by E. Reissner for the problem of elastic pure bending of thin-walled curvilinear tubes is considered. The formulation of the problem is refined, and the numerical solution of the equations obtained is compared with the finite-element results.

Introduction. One of the lines of the stability analysis of tubes under bending is based on taking into account the subcritical deformation (flattening) of cross sections. Brazier [1] studied the nonlinear deformation of a thin-walled cylindrical shell under pure bending and calculated the critical bending moment for which the instability occurs as a result of progressing flattening of cross sections of the shell. It was assumed in [1] that the flattening was constant along the tube. This made it possible to consider the problem in a one-dimensional formulation. The subsequent studies, which are reviewed, for example, in [2, 3], refine the results of [1] and develop the theory of finite bending of tubes with a curvilinear axis.

Reissner [4] reduced the problem of plane finite bending of a curvilinear thin-walled tube of an arbitrary cross section to a fourth-order system of differential equations supplemented by two integral conditions. To the author's knowledge, however, no solutions of these equations have been reported in the literature.

In the present paper, the formulation of the problem given in [4] is refined. For the case of a circular cross section, the effect of geometrical parameters of the tube is studied, and a comparison with the finite-element solution is performed.

Formulation of the Problem and Governing Equations. A thin-walled curvilinear tube with a cross-sectional radius b and a radius of the curvature of the axial line a is loaded in the plane of curvature by an external bending moment M (Fig. 1). In the deformed state, the tube is a toroidal shell with an unknown cross-sectional shape and radius of curvature of the axial line ρ. Since the cross section of the shell remains symmetric about the r axis, we consider one half of the cross section and seek for solution in the interval $-\pi/2 \leqslant \xi \leqslant \pi/2$.

According to [4], the system of governing equations comprises:

— strain compatibility equation

$$\frac{aB}{b^2}\Big[\Big(1+\frac{b}{a}\sin\xi\Big)\Psi'\Big]' = k\cos\xi + (1+k)[\cos(\beta+\xi)-\cos\xi];$$

— equation of equilibrium of bending moments

$$\frac{aD}{b^2}\Big[\Big(1+\frac{b}{a}\sin\xi\Big)\beta'\Big]' = (1+k)\Psi\sin(\beta+\xi) - C\cos(\beta+\xi),$$

the general solution of which should satisfy the boundary conditions $\Psi(\pm\pi/2)=0$ and $\beta(\pm\pi/2)=0$;

— integral conditions of equilibrium, which serve to determine the parameters ρ and C

$$-2b\int\limits_{-\pi/2}^{\pi/2}\Psi\cos(\beta+\xi)\,d\xi = M;$$

$$\int\limits_{-\pi/2}^{\pi/2}\Psi\sin(\beta+\xi)\,d\xi = 0. \qquad (1)$$

Chaplygin Siberian Aviation Institute, Novosibirsk 630051. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 42, No. 5, pp. 193–197, September–October, 2001. Original article submitted April 16, 2001.

0021-8944/01/4205-0898 $25.00

Here $\Psi(\xi)$ is the stress function, $\beta(\xi)$ is the angle of rotation of the tangent to the contour of the cross section, $k = a/\rho - 1$, B and D are the elastic coefficients, and C is an unknown parameter which has the meaning of the internal force acting in the z direction, the prime denotes derivatives with respect to the coordinate ξ.

Remark 1. In [4], condition (1) is obtained from the condition that the bending moment about the r axis vanishes

$$\oint \Psi \sin(\beta + \xi)\, d\xi = 0. \tag{2}$$

One can readily show that this condition is satisfied identically by virtue of the fact that the strain and stress distributions are symmetric about the r axis, and hence, it cannot be used as an additional condition. The replacement of the integral over the closed contour in (2) by the integral over the domain $-\pi/2 \leqslant \xi \leqslant \pi/2$ is not feasible, since the meaning of condition (2) is violated in this case. Therefore, instead of (1), one should use the condition

$$\int\limits_{-\pi/2}^{\pi/2} \sin(\beta + \xi)\, d\xi = 0,$$

which expresses the equality of the ordinates of the points of the section $\xi = -\pi/2$ and $\xi = \pi/2$.

We introduce the dimensionless quantities

$$\lambda = \frac{b}{a}, \quad \psi = \sqrt{\frac{B}{D}}\,\Psi, \quad \mu = \frac{b^2}{a\sqrt{BD}}, \quad \alpha = \Big(\frac{1}{\rho} - \frac{1}{a}\Big)\frac{b^2}{\sqrt{BD}}, \quad c = C\,\frac{b^2}{aD}, \quad m = \frac{M}{\pi b}\sqrt{\frac{B}{D}}. \tag{3}$$

With allowance for the Remark 1 given above and (3), the system for determining the functions ψ and β and the parameters α and c becomes

$$[(1 + \lambda \sin\xi)\psi']' - (\mu + \alpha)\cos(\beta + \xi) + \mu \cos\xi = 0; \tag{4}$$

$$[(1 + \lambda \sin\xi)\beta']' - (\mu + \alpha)\psi \sin(\beta + \xi) + c\cos(\beta + \xi) = 0; \tag{5}$$

$$\psi(\pm\pi/2) = 0, \qquad \beta(\pm\pi/2) = 0; \tag{6}$$

$$-\frac{\alpha}{\pi}\int\limits_{-\pi/2}^{\pi/2} \psi \cos(\beta + \xi)\, d\xi = m; \tag{7}$$

$$\int\limits_{-\pi/2}^{\pi/2} \sin(\beta + \xi)\, d\xi = 0. \tag{8}$$

Numerical Method for Solving the Problem. We use a step-by-step procedure for solving the nonlinear system (4)–(8) with iterative refinement of the solution by the Newton method. The parameter of the curvature change of the shell axial line α is used as a continuation parameter. Replacing the derivatives in (4) and (5) by finite differences and calculating the integral in (8) by the trapezoidal rule, we obtain the system

$$A_i\,\delta\psi_{i-1} - 2\,\delta\psi_i + B_i\,\delta\psi_{i+1} + C_i\,\delta\beta_i = D_i,$$

$$A_i\,\delta\beta_{i-1} - E_i\,\delta\beta_i + B_i\,\delta\beta_{i+1} - C_i\,\delta\psi_i + F_i\,\delta c = G_i \qquad (i = 1, \ldots, n-1),$$

$$\sum_{i=1}^{n-1} \cos(\xi_i + \beta_i)\,\delta\beta_i = -\sum_{i=1}^{n-1} \sin(\xi_i + \beta_i),$$

where

$$A_i = 1 - \frac{h}{2}\,\frac{\lambda\cos\xi_i}{1 + \lambda\sin\xi_i}, \quad B_i = 1 + \frac{h}{2}\,\frac{\lambda\cos\xi_i}{1 + \lambda\sin\xi_i}, \quad C_i = q_i(\mu + \alpha)\sin(\xi_i + \beta_i),$$

$$D_i = -A_i\psi_{i-1} + 2\psi_i - B_i\psi_{i+1} + q_i[(\mu + \alpha)\cos(\xi_i + \beta_i) - \mu\cos\xi_i],$$

TABLE 1

λ	$m_{\rm cr}$			$\delta_{\rm cr}$			$æ_{\rm cr}(0)$		
	$b/t = 25$	$b/t = 50$	$b/t = 100$	$b/t = 25$	$b/t = 50$	$b/t = 100$	$b/t = 25$	$b/t = 50$	$b/t = 100$
0.500	0.1642 (0.1703)	0.1195 (0.1226)	0.0862 (0.0877)	0.3139 (0.3326)	0.2879 (0.3011)	0.2692 (0.2769)	3.368 (3.542)	4.929 (5.145)	7.152 (7.343)
0.250	0.2229 (0.2249)	0.1642 (0.1653)	0.1195 (0.1200)	0.3484 (0.3525)	0.3138 (0.3168)	0.2880 (0.2894)	2.249 (2.245)	3.367 (3.383)	4.930 (4.941)
0.125	0.2979 (0.2984)	0.2228 (0.2232)	0.1639 (0.1643)	0.3980 (0.3987)	0.3483 (0.3489)	0.3138 (0.3139)	1.443 (1.428)	2.248 (2.242)	3.366 (3.359)

Note. The values in parenthesis are obtained by the finite-element method [3].

$$E_i = 2 + q_i[\psi_i(\mu + \alpha)\cos(\xi_i + \beta_i) + c\sin(\xi_i + \beta_i)],$$

$$F_i = q_i\cos(\xi_i + \beta_i), \quad G_i = -A_i\beta_{i-1} + 2\beta_i - B_i\beta_{i+1} + C_i\psi_i - F_i c, \quad q_i = h^2/(1 + \lambda\sin\xi_i),$$

and n is the number of equal segments of length h into which the interval $-\pi/2 \leqslant \xi \leqslant \pi/2$ is divided.

The boundary conditions have the form $\delta\beta_0 = \delta\beta_n = \delta\psi_0 = \delta\psi_n = 0$.

As the initial approximation, one can use the zero solution. In this case, the first iteration at the first step of the process yields the linear solution of the problem.

Calculation Results. We consider an isotropic material of the shell with Young's modulus E and Poisson's ratio ν. Using the expressions $B = 1/(Et)$ and $D = Et^3/(12(1-\nu^2))$ (t is the thickness of the shell) and (3), we write the parameter of the initial curvature of the tube in the form $\mu = \sqrt{12(1-\nu^2)}\,(b/t)\lambda$.

We study the effect of the geometrical parameters λ and b/t on the solution of Eqs. (4)–(8) for $\nu = 0.3$. We confine ourselves to the case where the external bending moment increases the curvature of the tube. Table 1 lists the following parameters of the critical state: the dimensionless bending moment m, the relative flattening displacement of the cross section $\delta = 1 - \dfrac{1}{2}\displaystyle\int\limits_{-\pi/2}^{\pi/2}\cos(\xi + \beta)\,d\xi$, and the change in the cross-sectional curvature $æ(0) = \beta'(0) - 1$ at the point $\xi = 0$. We note that the diagonal values of $m_{\rm cr}$, $\delta_{\rm cr}$, and $æ_{\rm cr}$ in Table 1 correspond to the same value of μ. This allows us to study the effect of the parameters λ and b/t corresponding to a fixed value of μ.

One can see from Table 1 that the solution of Eqs. (4)–(8) is close to the finite-element results. The most noticeable discrepancy, which does not exceed 6%, is observed for tubes with a large initial curvature and relatively thick wall. This discrepancy can be attributed to the fact that Eqs. (4)–(8) were derived under the assumption of zero meridional strain and circumferential bending moment.

An analysis of the data listed in Table 1 shows that the determining geometrical parameter of a curvilinear tube is μ, i.e., variation in λ and b/t for constant μ affects the results only slightly. For example, a decrease in λ from 0.5 to 0.125 for $\mu = 41.307$ leads to a decrease in the critical moment $m_{\rm cr}$ by only 0.02%. Therefore, the parameter λ in Eqs. (4) and (5) can be ignored. This conclusion is supported by the results of the numerical solution of Eqs. (4)–(8) obtained for $\lambda = 0$. These results coincide (with accuracy to the third decimal place) with those given in Table 1. It was found that $c = 0$ in this case. Consequently, one can use simplified equations obtained from Eqs. (4) and (5) if the terms with the factors λ and c are neglected and condition (8) is ignored. Precisely these equations were proposed in [5]; however, the approximate solutions given in [5] are of limited applicability.

Conclusions. The geometrically nonlinear formulation of the problem of pure bending of a thin-walled curvilinear tube proposed in [4] has been reconsidered. The system of governing equations has been refined by replacing the integral condition of equilibrium by a kinematic condition that stems from the symmetry of deformation of the cross section. The numerical solution of the equations agrees well with the finite-element results in a wide range of geometrical parameters of the tube. Taking into account the parameters λ and c in Eqs. (4)–(8) affects the calculation results only slightly. The main parameter that determines the initial geometry of the tube is μ.

REFERENCES

1. L. G. Brazier, "On the flexure of thin cylindrical shells and other "thin" sections," *Proc. Roy. Soc. London, Ser. A.*, **116**, 104–114 (1927).
2. J. Spence and S. L. Toh, "Collapse of thin orthotropic, elliptical cylindrical shells under combined bending and pressure loads," *Trans. ASME, Ser. E, J. Appl. Mech.*, **46**, 363–371 (1979).
3. V. V. Kuznetsov and S. V. Levyakov, "Nonlinear Kármán problem for toroidal shells of an arbitrary cross section," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 2, 136–142 (1992).
4. E. Reissner, "On the finite pure bending of curved tubes," *Int. J. Solids Structures*, **17**, 839–844 (1981).
5. É. L. Axelrad, "Bending of thin-walled rods in the region of large elastic displacements," *Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk.*, No. 3, 124–132 (1961).